

A Non-Classical Linear Xenomorph as a Model for Quantum Causal Space

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Abstract

A quantum picture of the causal structure of Minkowski space \mathcal{M} is presented. The mathematical model employed to this end is a non-classical version of the classical topos $\{\mathbb{H}\}$ of real quaternion algebras used elsewhere to organize the perceptions of spacetime events of a Boolean observer into \mathcal{M} . Certain key properties of this new quantum topos are highlighted by contrast against the corresponding ones of its classical counterpart $\{\mathbb{H}\}$ modelling \mathcal{M} and are seen to accord with some key features of the algebraically quantized causal set structure.

1 INTRODUCTION

Real four-dimensional Minkowski space \mathcal{M} is what a Boolean researcher perceives (Trifonov, 1995). Her perceptions of spacetime events, or equivalently, her states of knowledge of the world, that is to say, her controlling actions on and ‘passive’ observations of the events of the world, are uniquely modelled by the topos $\{\mathbb{H}\}$ of real quaternion division algebras¹ which are then seen to effectively encode in their multiplicative structure the Lorentzian metric

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¹A note on unusual language: in (Trifonov, 1995) the real quaternion objects \mathbb{H} of this Boolean topos are called \mathbb{R} -paradigms and the topos itself a classical \mathbb{R} -xenomorph. The general (classical) topos theory for \mathcal{M} may be coined (Classical) Linear Xenology (Trifonov in private e-mail correspondence).

$\eta_{\mu\nu} = \text{diag}(-1, +1, +1, +1)$ of \mathcal{M} . Since from the latter derives the causal structure at each spacetime event, the Minkowski lightcone soldered at each event, the result above can be rephrased as follows: a researcher that orders her perceptions of spacetime events according to classical Boolean logic, that is, a classical researcher, builds a unique picture of a special relativistic causal order between them and the latter is effectively what she ‘sees’ as the world’s chronological connection. \mathcal{M} as a causal space is what a classical researcher perceives. This explains the reality (\mathbb{R}), dimensionality (4) and signature (+2) characters of \mathcal{M} as inevitable results of the Boolean mode of perception (logic algebras) of spacetime events of a classical researcher and due to this \mathcal{M} may be called ‘classical causal space’.

On the other hand, it was established early (Robb, 1914, 1921, Alexandroff, 1956 and Zeeman, 1964, 1967) and revived lately (Sorkin *et al.*, 1987) that causality, when modelled by a partial order between events, determines as well the three characteristics of \mathcal{M} above². It follows that a partial order may be thought of as a sound model of the classical causality relation which is compatible or in accord with the Boolean mode of perception of spacetime events of a classical researcher. Since the latter is organized into the classical topos $\{\mathbb{H}\}$, we infer that the multiplicative structure of quaternions is intimately related to that of posets and both to a classical, Boolean logical perception of the event-structure of the world. Thus, it is fairly natural to suppose that a non-classical researcher, defined as one that employs some kind of non-Boolean or ‘quantal’ logic to order her perceptions of spacetime events, builds based on it a quantum picture of the causal structure of \mathcal{M} , call it ${}_q\mathcal{M}$. The latter may be equivalently viewed as the product of some sort of quantization of the classical causality relation which is represented by a partial order in \mathcal{M} .

In the present paper we present a candidate for such a quantum version of the causal structure of Minkowski space based on a non-classical model topos $\{\$ \}$ of a new algebraic structure $\$$ that is a multiplicative deformation of the quaternions and was introduced in (Raptis, 1998). We baptize the $\{\$ \}$ model of ${}_q\mathcal{M}$ ‘quantum topos’³ and study its properties in comparison

²The work of Sorkin *et al.* on a causal set theory of the small scale structure of spacetime goes even a bit further and argues that not only the Lorentzian metric signature, real coefficient field and the four-dimensionality of \mathcal{M} are determined when a partial order models causality between events, but also its topological and differential structure.

³Or, to comply with (Trifonov, 1995), ‘quantum linear xenomorph’ and the general

with those of its classical counterpart $\{\mathbb{H}\}$ model of \mathcal{M} ⁴. The \mathbb{H} -deformed multiplicative structure of $\$$ encodes a quantum sort of causality distinct from the classical Minkowskian one of \mathbb{H} which can be equivalently cast as a partial order as mentioned above. A recent result from a straightforward algebraic quantization of the classical partial order causality (Raptis, 1999) further supports the soundness of $\{\$$ as a non-classical linear xenomorph model of the quantum causal space ${}_q\mathcal{M}$.

The paper is organized as follows: in Section 2 we recall some basic terminology, essential facts and results from (Trifonov, 1995), mainly that a classical researcher ‘uniquely determines’ 4-dimensional, real Minkowski space as the structure of her own (proper) states of knowledge of the events of the world, the latter being organized into the classical linear \mathbb{R} -xenomorph $\{\mathbb{H}\}$. In Section 3 we briefly present how causality, modelled by a partial order between events, also ‘uniquely determines’ \mathcal{M} , hence infer that a classical researcher, by using her Boolean logic, builds a picture (model) of the chronological connection between her event-perceptions effectively isomorphic to a partial order, with the latter justly called ‘classical causality’. In the last Section 4 we borrow $\$$ from (Raptis, 1998) and organize the non-classical linear \mathbb{C} -xenomorph $\{\$$. After comparing it with $\{\mathbb{H}\}$ we suggest that it is what a ‘quantal researcher’ perceives as ${}_q\mathcal{M}$ having a quantum version of the classical causality relation of \mathcal{M} encoded in the multiplicative structure of its $\$$ -objects. This is a non-classical linear \mathbb{C} -xenomorph modelling the quantum causal space ${}_q\mathcal{M}$. We find that a quantal subobject classifier in the quantum topos $\{\$$ is the Lorentz-spin algebra $sl(2, \mathbb{C})$ that corresponds to the relativistic invariances of ${}_q\mathcal{M}$ and is the quantum substitute for the usual Boolean binary alternative **2**-the subobject classifier in the classical topos $\{\mathbb{H}\}$ model of \mathcal{M} . At the end we give some heuristic arguments to support that this $\{\$$ picture of ${}_q\mathcal{M}$ agrees with the algebraically quantized causal sets presented in (Raptis, 1999). In the Conclusion we give a résumé of the paper and comment briefly on a possible application of our quantum topos idea to the problem of quantum gravity.

quantum topos theory that may be developed from it ‘Quantum Linear Xenology’.

⁴This will go some, but still short, way in determining the ‘true quantum topos’ for quantum relativistic spacetime. The latter may prove to be an instance of the crucial missing denominator in the riddling analogy $\frac{\text{locales}}{\text{quantaes}} = \frac{\text{topoi}}{?}$ that has puzzled mathematicians for a while now (Lambek and Selesnick in private s-mail and e-mail correspondence).

2 THE CLASSICAL LINEAR XENOMORPH

In this section we resume some basic features of classical Linear Xenology from (Trifonov, 1995).

a. The elementary actions of a researcher effecting events upon the world form a semigroup with a two-sided identity; a monoid M . M is called the motor or effector space of the researcher. M models our primitive intuition that the series composition of two classical (selective) actions is such an action and that this composition is associative.

b. The observations of a researcher of the events of the world form a linear space S over a field \mathcal{F} ; a vector space S . S is called the sensor or reflexor space of the researcher. S models our primitive intuition that the classical sense-pictures of the world that a researcher perceives participate in incoherent, \mathcal{F} -weighed superpositions and that the latter too are the researcher's mixed, 'probability-weighed' sensors of the world.

c. The Quantum Principle, by positing that every sensor of a researcher is one of her motors, combines M and S into an associative algebra A over \mathcal{F} with an identity. Trifonov follows the constructivists' jargon and coins the elements of A the researcher's states of knowledge of the world, while A as a whole, the paradigm of (the structure of) the world constructed by the researcher.

d. The time proper to the researcher is defined in a constructivistic sense according to which it partially orders her states of knowledge. Thus, for a given paradigm A over an ordered field \mathcal{F} such as \mathbb{R} , time can be represented by a real 1-form τ on A 's sensor space S , that is, an \mathbb{R} -valued linear functional on A as a vector space.

e. There is a 'naturally' defined metric $g_{\mu\nu}$ for any given paradigm A once the proper time functional τ is fixed on it. $g_{\mu\nu}$ is induced by A 's structure constant 3-tensor $C(a; b, c)$ which is a trilinear functional on two vector arguments b, c in S and on one covector argument a in $S^{\star 5}$ with coordinates $C_{\mu\nu}^\lambda$ in \mathcal{F} . Once a is identified with τ in C and one insists that the latter be symmetric in the other two vector arguments, one is left with an \mathcal{F} -valued symmetric tensor $g_{\mu\nu}$ on $S \times S$.

f. The organization of isomorphic copies of A into a category $\{A\}$ of linear algebras over \mathcal{F} having the properties of a topos is called an \mathcal{F} -

⁵ S 's linear dual space.

xenomorph. An \mathcal{F} -xenomorph may be thought of as a universe of physico-mathematical discourse members of which, the researchers, based on their ‘research parameters of reasoning and measurement’, that is, their logic and \mathcal{F} -coordinatization, construct a model of the world according to their interactions with (research on) it.

g. An \mathcal{F} -xenomorph is of (finite) dimension n when its paradigms are n -dimensional vector spaces. If the motor space M of A is (not) a monoid, A is said to be (ir)rational. An \mathcal{F} -xenomorph of rational paradigms which is (not) Boolean is called (non)-classical. When a classical \mathcal{F} -xenomorph consists of paradigms of finite dimensionality, it is called classic.

h. The basic result from (Trifonov, 1995) is that when the research parameters are fixed to their current ‘values’, the latter being Boolean logic for the researcher and \mathbb{R} -coefficient field for her psychology, the only classic paradigm A ‘compatible’ with them is the 4-dimensional associative division algebra of real quaternions \mathbb{H} .

By fixing τ in \mathbb{H} ’s structure constant tensor C and symmetrizing with respect to the other two vector arguments in it as explained in e, one extracts the Minkowski metric $\eta_{\mu\nu} = \text{diag}(+1, -1, -1, -1)$ of Lorentzian signature $|\text{tr}(\eta)| = 2$ directly from the multiplication table of the unit quaternions. Thus the motor structure of the classic paradigm \mathbb{H} determines the special relativistic chronometric $\eta_{\mu\nu}$ on $S_{\mathbb{H}}$ and the latter is then identified with Minkowski space \mathcal{M} . Moreover, once a classical researcher fixes her (non-relativistic) proper time τ in $C_{\mathbb{H}}$, her motor structure induces in turn the special relativistic chronometric delimiting the invariant causal structure on her sensor space (the Minkowski lightcone). Thus proper time, in the constructivist sense of a partial order on the researcher’s states of knowledge (d), which is a non-physical mental attribute, generates via the motor structure of the classical researcher’s paradigm \mathbb{H} the physical metric on \mathcal{M} .

The Lorentz transformations of special relativity arise naturally from the categorical arrows between the quaternion paradigms in the topos $\{\mathbb{H}\}$ that, by their definition as linear homomorphisms, preserve the linear (sensory) and multiplicative (motor) structure of the \mathbb{H} objects in $\{\mathbb{H}\}$ (ie they preserve \mathcal{M} as a vector space and its lightcone induced by \mathbb{H} ’s multiplicative structure). We wrap-up this to the following: a Boolean researcher with real proper time as a partial order on her states of knowledge of the world uniquely describes the latter as \mathcal{M} with its special relativistic causal structure.

i. The concluding remark from (Trifonov, 1995) that we mention here is

that a non-trivial Grassmannian or supersymmetric paradigm \mathcal{G} , one having non-trivial divisors of zero in its motor space M , is proper to a non-Boolean researcher and her topos $\{\mathcal{G}\}$ is non-classical. Finkelstein (1996) and Selesnick (1998) have suggested *Qet*, a certain version of the category $\{\mathcal{G}\}$ of Grassmann algebras, as a quantum replacement of the classical topos *Set* of sets. *Qet* models quantum set theory as *Set* is the universe of classical set theory and the basic insight of Finkelstein is that since classical spacetime is primarily a classical set soundly represented in the Boolean topos *Set*, quantum spacetime must first of all be a quantum set living in the non-classical or ‘quantal’ linear xenomorph $\{\mathcal{G}\}$ ⁶. The work of Trifonov (1995) made the subtle contribution that relativistic spacetime \mathcal{M} is indeed a conception of a classical researcher, a classical entity living in a Boolean topos; however, not in *Set*, but in $\{\mathbb{H}\}$. Of course, if we neglect the algebraic structure from the objects of $\{\mathbb{H}\}$, from which the metric structure of \mathcal{M} derives as described in e and h above, there is a natural equivalence (functor) between it and *Set*. Since Finkelstein foremost intended *Qet* as a universe of discourse in quantum spacetime topology and not in quantum spacetime metric structure, it serves as a sound model for the former not the latter⁷. For the latter we expect a quantum replacement of $\{\mathbb{H}\}$ (not *Set*) to serve us as a model of ${}_q\mathcal{M}$: a quantum Minkowski spacetime.

We provide such a candidate quantum topos in Section 4. First we relate the $\{\mathbb{H}\}$ picture of \mathcal{M} above to its causal set one of Sorkin *et al.* (1987).

3 \mathcal{M} AS A CLASSICAL CAUSAL SPACE

In d of Section 2 we mentioned Trifonov’s assumption of a constructivistic notion of time according to which it partially orders the researcher’s states

⁶Finkelstein’s original intuition that quantum set theory should be modelled in the world *Qet* or $\{\mathcal{G}\}$ appeared in the literature as early as (1972a, b) and it is all resumed in Finkelstein (1996). Significant allusions to the quest for a quantum topos modelling quantum spacetime can also be found in (Selesnick, 1991, 1998) which is along the lines of Finkelstein’s work. To this end Selesnick too anticipates a quantum replacement of the topos *Set* by *Qet* or some other structure in order to accommodate quantum spacetime.

⁷Afterall, extra structure is usually imposed on a topological space, such as the Euclidean manifold \mathcal{M} one, to become a metric space. $\{\mathbb{H}\}$, stripped-off its algebraic structure which encodes $\eta_{\mu\nu}$, becomes a topos of classical sets the Boolean algebra of which suffices for defining a classical topological space-‘a collection of open sets such that...’.

of knowledge of the world. Thus time is not a physical property of the world *per se*, a connection between its physical events, but, rather indirectly, it is one between the researcher's perceptions of the events of the world. That this psychological conception of time is assumed to be a partial order however, accords with one of the most successful models for \mathcal{M} that has been constructed so far: the causal set of Sorkin *et al.* (1987). The causal set pictures spacetime as a locally finite set of events with the causal 'after' relation between them represented by a partial order (a locally finite poset). Sorkin and his coworkers hold that a partial order determines not only the metric-signature, reality and four-dimensionality of \mathcal{M} , but also its topological and differential structure; furthermore, due to its local finiteness, they maintain that it can serve as a discrete substratum underlying spacetime at short distances of the order of Planck's ($10^{-33}cm$) where quantum gravitational effects are expected to be significant.

This application in the context of quantum gravity aside, for the purposes of the present paper we mention the following affinity between Classical Linear Xenology and Causal Set Theory in the form of a conjecture: since \mathcal{M} is what a Boolean researcher with a real psychological proper time uniquely perceives as the event-structure of the world, and since the same spacetime is uniquely determined by assuming the causal structure of the events of the world to be a poset, a partial order is a classical conception of causality intimately related to a Boolean logical perception of the events of the world and their chronological connection. A causal set is a classical and classic causal space in the sense of g in Section 2. It follows that a non-classical researcher, one that uses a quantal sort of logic and the usual coefficient field \mathbb{C} for the quantum dynamical amplitudes, will construct a quantum version of \mathcal{M} , call it ${}_q\mathcal{M}$. This will be effectively a quantum replacement of the classical topos $\{\mathbb{H}\}$ model for \mathcal{M} with the resulting quantum topos serving as a model of quantum causal Minkowski space ${}_q\mathcal{M}$. Many noteworthy attempts have been made in the past at a straightforward quantum substitution of a classical causal space of which we mention Finkelstein's spacetime code (1969), superconducting causal nets (1988) and hypercubical quantum causal network (1996), as well as this author's more recent try (Raptis, 1999). In the following section, to realize our conjecture above, we suggest a non-classical topos as a model of the quantum causal space ${}_q\mathcal{M}$.

4 A NON-CLASSICAL TOPOS MODEL OF ${}_q\mathcal{M}$

Below we introduce a new algebraic paradigm $\$$ and interpret it as the local spacetime perceptions of a non-classical researcher. Then we ‘globalize’ the result by organizing such objects into a \mathbb{C} -xenomorph, a quantum topos $\{\$\}$, and maintain that it models ${}_q\mathcal{M}$ as a quantum causal space. Because a full-fledged presentation of $\$$ will take much space thus perhaps disorientate our focus, we concentrate on the formal properties of $\$$ that directly compare with those of the classic paradigm \mathbb{H} for \mathcal{M} in Section 2. This will highlight $\{\$\}$ ’s quantal properties in contradistinction to $\{\mathbb{H}\}$ ’s classical ones. For more on $\$$ the reader may refer to the analytic treatment of it in (Raptis, 1998).

a’. The linear, sensor structure of $\$, S_\$,$ is that of a 4-dimensional vector space over the complex field \mathbb{C} which is the usual coefficient field of quantum mechanics. As in (Trifonov, 1995) the standard quaternion basis $\{1, i, j, k\}$ is assumed to be the reference frame proper to a classical researcher, so in $\$$ the standard basis is $g = \{I, \imath, \sigma, \tau\}$ ⁸. g is supposed to be the reference frame proper to a quantal researcher. Unlike the real psychology \mathbb{R} of the quaternion paradigm of a classical researcher to which her psychological time takes its values, orders her states of knowledge of the world and generates the local causal structure of \mathcal{M} as in e of Section 2, the complex psychology \mathbb{C} of a quantal researcher must radically change her conception of the structure of quantum spacetime. For instance, the principal difference between the sensor structure of $\$$ and that of \mathbb{H} is that the linear superpositions over \mathbb{C} in $S_\$$ are coherent, that is, quantum amplitude \mathbb{C} -weighed ones, while those in $S_{\mathbb{H}}$ over \mathbb{R} are incoherent, that is, classical probability \mathbb{R} -weighed ones as mentioned in b of Section 2. The difference between coherent and incoherent superpositions is a fundamental one that essentially distinguishes quantum from classical algebraic structures (Finkelstein, 1996). For this, ${}_q\mathcal{M}$ as a vector space, whose quantum paradigm is $\$,$ is isomorphic to \mathbb{C}^4 rather than to \mathbb{R}^4 which is the linear structure of the classical \mathbb{H} paradigm of \mathcal{M} . Furthermore, there is a canonical complex model for quantum Minkowski space associated with \mathbb{C}^4 , namely, the Grassmannian space $\mathcal{G}_2(\mathbb{C}^4)$ of two dimensional subspaces of \mathbb{C}^4 (Selesnick, 1991). The Grassmannian (supersymmetric) nature of the quantal paradigm $\$$ for ${}_q\mathcal{M}$ is presented in d’ below.

⁸The vector τ in g should not be confused with the proper time τ in d of Section 2.

b'. The multiplicative, motor structure of $\$, M_\$,$ can be resumed in the following table $T(\$)$ showing the binary product between the elements of g^9

$$T(\$) = \begin{array}{c|c|c|c|c} \$ & I & \imath & \sigma & \tau \\ \hline I & I & \imath & -\tau & -\sigma \\ \hline \imath & \imath & -I & -\tau & \sigma \\ \hline \sigma & \sigma & \tau & I & -\imath \\ \hline \tau & \tau & -\sigma & \imath & -I \end{array} .$$

Direct inspection of the table shows that the product in $\$$ is not associative so that its motor structure is not a semigroup. Also, I is a right but not a left identity, so that $M_\$$ is *a fortiori* not a monoid. In view of the proof of the basic theorem in (Trifonov, 1995) and h in Section 2, since a quantal researcher employs a non-Boolean, quantal logic and a complex psychology to order her states of knowledge of the world, the algebraic paradigm of the structure of the world that she builds based on them is expected to be not only non-classical, but also irrational.

From a classical perspective, the ‘irrationality’ of quantal (selective) actions has already been observed by Finkelstein (1996) and Selesnick (1998) and can be resumed in the following: the series composition of two proper (selective) actions may not be a proper (selective) action. This is a principal characteristic of quantal actions and, as Finkelstein and Selesnick argue, it is equivalent to non-commuting proper actions, which is also due to their coherent superpositions mentioned in a’ above, which in turn make the logic of the researcher quantum¹⁰. This further qualifies $\$$ as a prime candidate quantal paradigm to model ${}_q\mathcal{M}$. At the same time however, the product between proper actions in $\$$ seems to violate not only their algebraic closure in it (ie the series product of two actions in $\$$ is not a proper action in $\$$),

⁹From now on, due to their multiplicative serial compositions, we call the basis elements in g $\$$ ’s generators. The generators of $\$$ are interpreted as elementary actions proper to the quantal experimenter who builds $\$$ as a paradigm-model of the structure of the world.

¹⁰Quantum logic is a non-classical, that is, non-Boolean, logic exactly due to the coherent superpositions of quanta. In its lattice representations, quantum logic is modelled after non-distributive lattices, while Boolean lattices are always distributive. Non-distributivity is due to non-commutativity which is due to coherent superpositions of quantum actions.

or even the existence of inverses¹¹, but also the associative law. Finkelstein notes and asks in (1996):

...Our selective acts for the quantum do not all commute. It follows that the composition of two selective acts in series is not always a selective act...On the other hand, the associative law $A(BC) = (AB)C$ seems to persist. Can you see its empirical meaning ? What experiment would break it ? ...Because the concatenation of two selective acts is a more general kind of action, it is artificial and clumsy to separate logic from dynamics in quantum theory as we do in classical thought...

Since the associative motor structure of the classic paradigm \mathbb{H} encodes the (local) causal structure of \mathcal{M} ¹², we expect the non-associativity of the product in $\$$ to capture an essential trait of (local) ‘quantum causality’. In g' we give, along this ‘logical-motor-causal’ line of thought, a plausible answer to Finkelstein’s question on the physical meaning of non-associativity and in the Conclusion we address his ‘*logics comes from dynamics*’¹³ in the opposite way: ‘(quantum) causality comes from (quantum) logics’.

c' . Like in d and e of Section 2, in $\$$ too we identify a proper time linear functional τ on $S_\$$ taking its values in \mathbb{C} and such that, when identified with a and evaluated in the structure constant tensor $C(a; b, c)$ of $\$$, it induces a (quantum) spacetime metric proper to the paradigm. Let us recall in more detail from (Trifonov, 1995) how this is done.

For the classical paradigm \mathbb{H} of \mathcal{M} the components $g_{\mu\nu}$ of the metric in some basis $\{e_\mu\}$ and its dual $\{e^\nu\}$ are given by

$$g_{\mu\nu} = C(\tau; b, c) = C(\tau_\lambda e^\lambda; b^\mu e_\mu, c^\nu e_\nu) = \tau_\lambda C_{\mu\nu}^\lambda .$$

Then, in the standard basis $\{1, i, j, k\}$ for \mathbb{H} $g_{\mu\nu}$ reads

¹¹ $\$, unlike the classical quaternion paradigm, can easily be shown to be not a division ring. In (Trifonov, 1995) it is argued that if the logic of the researcher is Boolean the motor structure of her proper paradigm of the world is a group. For \mathbb{H} in particular, $M_{\mathbb{H}}$ is the multiplicative group of non-zero quaternions isomorphic to $SU(2) \times \mathbb{R}^+$. A non-classical, quantal logic for the researcher is consistent with non-invertible motor actions on the world. We return to some implications of this remark in the Conclusion at the end.$

¹²Section 2e, also witness in c' that follows how the local Lorentzian chronometric $\eta_{\mu\nu}$ is effectively encoded in the binary product between the generators of \mathbb{H} .

¹³From the Foreword ‘About the Philosophy’ in (Finkelstein, 1996).

| | | | |
|----------|-----------|-----------|-----------|
| τ_0 | τ_1 | τ_2 | τ_3 |
| τ_1 | $-\tau_0$ | $-\tau_3$ | τ_2 |
| τ_2 | τ_3 | $-\tau_0$ | $-\tau_1$ |
| τ_3 | $-\tau_2$ | τ_1 | $-\tau_0$ |

Notice that it effectively corresponds to the product table $T(\mathbb{H})$ of the standard unit quaternions multiplied by the components τ_μ of the proper time in the dual standard basis for \mathbb{H} . To see this just identify $\tau_0 \equiv 1$, $\tau_1 \equiv i$, $\tau_2 \equiv j$ and $\tau_3 \equiv k$ in the table above and get the entries in $T(\mathbb{H})$ below

$$T(\mathbb{H}) = \begin{array}{c|cccc} \mathbb{H} & 1 & i & j & k \\ \hline 1 & 1 & i & j & k \\ i & i & -1 & -k & j \\ j & j & k & -1 & -i \\ k & k & -j & i & -1 \end{array} \text{ with } kji = -1.$$

By insisting that $g_{\mu\nu}$ is a symmetric matrix we get $\tau_1 = \tau_2 = \tau_3 = 0$ and we recover, up to a scalar factor τ_0 , the Lorentz metric $\eta_{\mu\nu} = \text{diag}(1, -1, -1, -1)$ with trace of absolute value 2. In this sense ‘*if we ignore the motor structure of the paradigm, four-dimensionality and Lorentz metric become a mystery*’ (Trifonov, 1995). The Lorentz metric is effectively encoded in the binary multiplication table of the unit quaternions. Also notice that $\eta_{\mu\nu}$ is the component $C_{\mu\mu}^0$ of the structure constant tensor $C(\mathbb{H})$ and it is generated by the ‘wrist-watch’ psychological time $(\tau_0, 0, 0, 0)$. In the product table of the quaternions this is the ± 1 entries along the main diagonal corresponding to the squares of the four unit quaternions.

Similarly, for the quantal paradigm $\$$ of ${}_q\mathcal{M}$ the components of the metric in the standard basis g are derived as for $\eta_{\mu\nu}$ in \mathbb{H} above so that, after symmetrization and along τ_0 , we read from $T(\$)$ the diagonal metric matrix $\kappa_{\mu\nu} = \text{diag}(1, -1, 1, -1)$ of signature 0. This is the traceless Klein metric. The physical significance of this form of the metric for ${}_q\mathcal{M}$ will be given shortly when we discuss the special relativistic and causal symmetries of $\$$ associated with $\kappa_{\mu\nu}$. First we have to show the Grassmannian or supersymmetric character of $\$$ as promised at the end of a’.

d’. The sensor space $S_\$$ is \mathbf{Z}_2 -graded as follows

$$S_{\$} = S_{\$}^0 \oplus S_{\$}^1 = \text{span}_{\mathbb{C}}\{I, \iota\} \oplus \text{span}_{\mathbb{C}}\{\sigma, \tau\}$$

with $S_{\0 , spanned by I and ι , the even, complex, 2-dimensional subspace of \mathbb{C}^4 of grade 0 elements of $\$$ and $S_{\1 , spanned by σ and τ , the odd, complex, 2-dimensional subspace of \mathbb{C}^4 of grade 1 elements of $\$$. This makes $\$$ a Lie superalgebra (Raptis, 1998), thus it is a Grassmannian or supersymmetric paradigm in the sense of i of Section 2. The $(2, 2)$ split of $S_{\$} \simeq \mathbb{C}^4$ by grade effects a natural isomorphism between $\$$ and the Grassmannian paradigm $\mathcal{G}_2(\mathbb{C}^4)$ of two dimensional subspaces of complex Minkowski space used by Selesnick (1991) to model ${}_q\mathcal{M}$ and alluded to at the end of a'. Thus $\$$ qualifies even further as a sound model of ${}_q\mathcal{M}$.

The physical interpretation of members of $S_{\0 is bosons and of those in $S_{\1 fermions. We may define the parity (or grade) binary alternative $\pi(x)$ by

$$\pi(x) := \begin{cases} 0 & \text{if } x \in S_{\$}^0, \\ 1 & \text{if } x \in S_{\$}^1, \end{cases}$$

and tentatively interpret the direct sum (\oplus) $(2, 2)$ split of $S_{\$}$ above as the usual SUSY Wick-Wightman-Wigner spin-statistics superselection rule that forbids the coherent quantum superpositions of bosons with fermions (Freund, 1986)¹⁴. However, due to our earlier interpretation in a' of $+$ as coherent quantum superposition throughout all $S_{\$}$, we anticipate a lifting of the spin-statistics superselection rule with concomitant formal replacement of the incoherent direct sum \oplus by coherent quantum superposition $+$. In $\$$ bosons are allowed to superpose with fermions¹⁵.

Since $\$$ is a Grassmannian paradigm it has non-trivial divisors of zero¹⁶, thus $M_{\$}$, unlike $M_{\mathbb{H}} \simeq SU(2) \times \mathbb{R}^+$, is not a group and its logic is non-

¹⁴Some authors like Freund (1986) write $V = V^0 \cup V^1$ for this incoherent \mathbb{Z}_2 split by grade of a vector space V into its even and odd subspaces, others like Okubo (1997) write $V = V^0 \oplus V^1$ as we do.

¹⁵The physical consequences for the quantum structure of spacetime of such a lifting of the spin-statistics superselection rule are currently under investigation by this author. For instance, since the spin-statistics connection and the spin-statistics superselection rule are 'theorems' that derive from the 'causal axiom' of Einstein Locality in quantum field theory on \mathcal{M} (Haag, 1992), a quantum causal theory for ${}_q\mathcal{M}$ perhaps makes the latter a non-valid assumption so that the spin-statistics superselection rule is violated.

¹⁶For example, $\tau + \sigma \in S_{\1 and $(\tau + \sigma)^2 = 0$.

Boolean (Trifonov, 1995; see also footnote 11 in b'). We emphasize again, $\$$ is a paradigm for ${}_q\mathcal{M}$ of a quantal researcher.

e'. We organize the quantal paradigms $\$$ into the non-classical \mathbb{C} -xenomorph $\{\$$. $\{\$$ qualifies as a quantum topos model of ${}_q\mathcal{M}$ (Raptis, 1998, Selesnick, 1998). A major difference in structure between the quantum topos $\{\$$ for ${}_q\mathcal{M}$ and its classical counterpart $\{\mathbb{H}\}$ for \mathcal{M} is one concerning their subobject classifiers Ω . $\{\mathbb{H}\}$, being a classical topos, has $\Omega = \mathbf{2} = \{0, 1\}$, the set of Boolean truth values, as subobject classifier, while $\{\$$, being $\{\mathbb{H}\}$'s quantal analogue, is expected to have a quantum version of $\mathbf{2}$ as subobject classifier. Selesnick (1994, 1998), working on Finkelstein's quantum set theory, net dynamics and relativity, found that a plausible candidate for such a quantum version of the classical binary alternative $\mathbf{2}$ of the Boolean topos *Set* is $SL(2, \mathbb{C})$, the Lorentz-spin group of special relativity.

For a technical definition of the subobject classifier the reader is referred to (Goldblatt, 1984)¹⁷. Roughly, and deriving from the classical topos *Set* of sets, the subobject classifier, tells one whether an object in the topos is 'smaller than' (included in, injected into, a subobject of) another or not. Thus its values are in some sense the 'inclusion symmetries' of the objects in the topos, as for example in *Set*, whether a set is included in another ($\Omega = 1$) or not ($\Omega = 0$). A familiar use of $\mathbf{2}$ in *Set* is that for every set α its power set $\mathbf{2}^\alpha$ is the one containing all the subobjects (subsets) of α . Then, in a subtle sense, the quantal binary symmetries $SL(2, \mathbb{C})$ of quantum spacetime modelled after the world *Qet* $\simeq \{\mathcal{G}\}$ of quantum sets, is the invariance group of the causal structure of special relativistic spacetime \mathcal{M} as perceived by a macroscopic, classical observer (Selesnick, 1994, 1998). The analogy between *Set* and *Qet* in this respect is the following: the inclusion $\beta \in \alpha$ or the injection $\beta \rightarrow \alpha$ between classical sets in *Set* practically corresponds to the causal connection $\beta \rightarrow \alpha$ between quantum spacetime events in *Qet*, so that $SL(2, \mathbb{C})$, like $\mathbf{2}$ in *Set*, is the invariance group of the past lightcone $\lambda(\alpha) := \{\beta : \beta \rightarrow \alpha\}$ of a quantum spacetime event α in *Qet*¹⁸.

Still though, as Selesnick (1994, 1998) emphasizes, the group $SL(2, \mathbb{C})$,

¹⁷For an excellent treatment of categories and topoi from a (classical) logical point of view the reader is also referred to (Lambek and Scott, 1986). A good treatment of topoi from a local set theoretic viewpoint is (Bell, 1988).

¹⁸For instance, for the quantum causal nets in (Finkelstein, 1988), $SL(2, \mathbb{C})$ is held to correspond precisely to the symmetry group acting on the two spinor inputs of the binary quantum causal cell.

the coherent state exponential of the Lie algebra $sl(2, \mathbb{C})$ of the quantum binary alternative, is what a coarse, macroscopic, classical observer with a limited power of resolution perceives as the invariance group of the underlying quantum Minkowski plenum ${}_q\mathcal{M}$. Our quantum topos $\{\$$ by comparison models the states of knowledge of ${}_q\mathcal{M}$ of a fine, ‘gedanken microscopic’, quantal researcher who is supposed to possess higher power of resolution than her classical counterpart¹⁹, thus operate at the quantum (algebra) not the classical (coherent state, group) level. Below we suggest an entirely discrete-algebraic (combinatorial) scheme, solely at the algebra level of $\$$, that leads directly to the Lorentz-spin structure $sl(2, \mathbb{C})$, which then serves as the quantal subobject classifier in the quantum topos $\{\$$ model of ${}_q\mathcal{M}$. This scheme is mainly based on quantum topos ideas that first appeared and were analytically treated in (Raptis, 1998).

f'. To derive the Lorentz-spin algebra $sl(2, \mathbb{C})$ as the causal symmetries of the quantum topos $\{\$$ for ${}_q\mathcal{M}$, we look at the two $(2, 2)$ splits of $S_{\mathbb{H}}$ by norm in c' and parity in d' . The first pertains to the $\mu = \pm 1$ signature binary distinction of the four generators of $\$$ in the Klein metric $\kappa_{\mu\nu} = \text{diag}(+1, -1, +1, -1)$, while the second to their $(-1)^{\pi(x)} = \pm 1$ grade binary distinction which we still call π . These two binary characteristics of the four generators of $\$$ are resumed in the following table

| $\mathbf{2 \times 2}$ | μ | π |
|-----------------------|-------|-------|
| I | $+1$ | $+1$ |
| \imath | -1 | $+1$ |
| σ | $+1$ | -1 |
| τ | -1 | -1 |

Then we assume that the Klein four-group $\mathbf{4}_2 = \mathbf{2} \times \mathbf{2}$ is the permutation symmetry group of the four generators of $\$$. By indexing the latter in g as $I = g_0$, $\imath = g_1$, $\sigma = g_2$ and $\tau = g_3$, the Klein four-group $\mathbf{4}_2 = \{(0)(1)(2)(3), (01)(23), (03)(14), (04)(13)\}$ permutes these indices and transposes their corresponding generators, of norm (μ) and parity (π) characteristics as in the table above, as follows

¹⁹For instance, as we have seen she is ideally assumed to perceive coherent quantum superpositions in her sensory space S , while her classical counterpart is limited to ‘sense’ only incoherent superpositions.

$$\begin{aligned}
(0)(1)(2)(3) &= \{I, \iota, \sigma, \tau\} = \text{'no permutation' (identity)}, \\
(01)(23) &= \{I \leftrightarrow \iota, \sigma \leftrightarrow \tau\} = \text{'norm swap at constant parity'}, \\
(02)(13) &= \{I \leftrightarrow \sigma, \iota \leftrightarrow \tau\} = \text{'parity swap at constant norm'}, \\
(03)(12) &= \{I \leftrightarrow \tau, \iota \leftrightarrow \sigma\} = \text{'norm and parity swap'}.
\end{aligned}$$

One may think of the norm and parity binary values of the g_μ s as their ‘causal colors’ and the $\mathbf{4}_2$ acting on them as their ‘causal color group’. Straightforward quantization of $\mathbf{4}_2$ à-la Finkelstein (1996) or Selesnick (1998) yields the Lorentz-spin algebra $sl(2, \mathbb{C})$, the ‘quantum causal color symmetry structure’, which a fine, quantal researcher perceives as the relativistic, causal symmetries of the quantum topos $\{\$$ model of ${}_q\mathcal{M}$ -its quantal subobject classifier. The whole dynamical or causal algebraic scheme in the quantum topos $\{\$$ may then be called ‘quantum causal chromodynamics’. From the same works (Finkelstein, 1996, Selesnick, 1998) it follows that a coarse, macroscopic, classical observer perceives coherent exponentials of $sl(2, \mathbb{C})$ which correspond to elements of the group $SL(2, \mathbb{C})$ -the quantal version of the classical subobject classifier $\mathbf{2}$ of the Boolean topos $\{\mathbb{H}\}$ model of \mathcal{M} .

In closing f’ we mention an affinity between our $\{\$$ model of ${}_q\mathcal{M}$ as a (quantal) causal space and Finkelstein’s null tesseract (quantum) causal net ${}_q\mathbb{N}^4$ (Finkelstein, 1996). The discrete causal symmetries of the latter constitute the symmetric group S_4 of $4! = 24$ permutations that act on a frame $\{n_\mu\}$ consisting of four linearly independent null vectors (null frame) and leave the following symmetric null form $n_{\mu\nu}$ of their mutual inner products invariant

$$n_{\mu\nu} = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}.$$

S_4 can be semi-factorized as $\mathbf{4}_2 \otimes \mathbf{3} \otimes \mathbf{2} \otimes \mathbf{1}$ with $\mathbf{4}_2$ the discrete version of the Lorentz-spin spacetime group as above, $\mathbf{3}$ the discrete ancestor of the GUT color gauge group $SU(3)$ and $\mathbf{2} \otimes \mathbf{1}$ the discrete precursor of the GUT electroweak gauge group $SU(2) \times U(1)$. The S_4 -symmetric null frame and form for \mathcal{M} is preferred over the Minkowskian $1+3$ ($1 + \{i, j, k\}$) quaternion frame and Lorentz metric $\eta_{\mu\nu}$ having only S_3 , the six permutations of its 3

unphysical spacelike vectors, as causal symmetries. Thus, even the internal, gauge symmetries are accounted for in an ‘external’, causal way, something that is not possible in the quaternionic, Lorentzian $1 + 3$ picture of \mathcal{M} .

In $\$$ on the other hand, $\kappa_{\mu\nu}$ is traceless like $n_{\mu\nu}$ and $\mathbf{4}_2$ also stands for the discrete, classically thought of as external, spacetime symmetries of \mathcal{M} , albeit, of a complexified version of it (\mathbb{C}^4). We found that in $\$$ even these external spacetime symmetries arise from permuting the ‘(quantum) causal colors’ of its generators. The latter are characteristic physical properties of $\$$ ’s generators that are intimately related to quantum integral/half-integral spin (parity $\pi = +1/-1$) and to relativistic spacelike/timelike (norm $\mu = +1/-1$) binary distinctions, respectively. Could it be that spacetime itself arises from some deep inner quantum relativistic algebraic distinctions such as these?²⁰ If yes, how is the relativistic ‘spacelike’ ($\mu = +1$) character related to the quantum ‘bosonic’ ($\pi = +1$) one and the timelike ($\mu = -1$) to the fermionic ($\pi = -1$)? Furthermore, while in (Raptis, 1998) an attempt to answer to these questions is made, it is also shown that one is able to form a null-frame like $\{n_\mu\}$ solely from algebraic associations of the generators of $\$$ and then straightforwardly apply Finkelstein’s S_4 ideas above.

g’. From a' to f' we arrived at the quantum topos $\{\$ \}$ as a cogent model of the quantum causal space ${}_q\mathcal{M}$ having as quantum causal symmetry structure the quantal subobject classifier $sl(2, \mathbb{C})$. The latter’s coherent form $SL(2, \mathbb{C})$, the double cover of the Lorentz group, is the quantum version of the Boolean binary alternative $\mathbf{2}$ -the subobject classifier of the classic xenomorph $\{\mathbb{H}\}$. The classical researcher building the $\{\mathbb{H}\}$ model of \mathcal{M} perceives $SL(2, \mathbb{C})$ in a ‘sensory-motor’ way (Trifonov, 1995) as the linear isometries relating various quaternionic frames e_μ of \mathbb{H} . It follows from Section 3 that while $\{\mathbb{H}\}$ is a model of classical causal space \mathcal{M} equivalent to a partial order between events and coined ‘classical causality’, $\{\$ \}$ is a model of quantum causal space ${}_q\mathcal{M}$ with an order relation between events other than a partial order.

The inadequacy of a partial order to model ‘quantum causality’ has been noticed by Finkelstein (1988, 1996) and recently exposed by this author (Raptis, 1999) in an attempt to algebraically quantize Sorkin *et al.*’s causal sets (1987). It was also explicitly anticipated in (Raptis, 1998). Briefly, a partial order fails to model ‘quantum causality’ on grounds of locality: because a

²⁰For a deep treatment of the notion of ‘distinction’ and how it may give rise to algebraic (spacetime) structures the reader is referred to (Kauffman, 1991).

partial order is transitive, thus mediated, it is not local, when in fact we suppose that all the fundamental (quantum) variables in Nature are local. Hence, we posit that ‘quantum causality’ is an intransitive, thus immediate, relation between events that should be algebraically represented. Such a model of ‘quantum causality’ was presented in (Raptis, 1999). Here we also propose an algebraic model²¹, a paradigm of ${}_q\mathcal{M}$ as a quantum causal space, thus it is natural to ask which algebraic feature of \mathbb{S} corresponds to the desirable intransitivity or immediacy of ‘quantum causality’ that it aspires to encode in its motor structure.

We pick the argument from the end of b’ and suggest that it is the non-associativity of the binary product in \mathbb{S} that corresponds to the intransitivity, for locality’s sake, of the quantum causal connection in ${}_q\mathcal{M}$ that \mathbb{S} algebraically models. We show this by the following heuristic argument which, while non-rigorous, is rather intuitive and ostensive: let ba , the series concatenation of two generators in \mathbb{S} from right to left, or equivalently, the series composition of two actions of the researcher in her motor space, stand for an immediate causal link $b \leftarrow a$. Then, the (in)transitivity of ‘ \leftarrow ’ can be symbolically, algebraically and equationally cast as (non)associativity

$$\begin{aligned} [(c \leftarrow b) \wedge (b \leftarrow a) \Rightarrow (c \leftarrow a)] &\Leftrightarrow [(cb)a = c(ba)] , \\ [(c \leftarrow b) \wedge (b \leftarrow a) \not\Rightarrow (c \leftarrow a)] &\Leftrightarrow [(cb)a \neq c(ba)] . \end{aligned}$$

It follows that the classical, associative quaternion paradigm \mathbb{H} is a sound algebraic model of the ‘classical causal space’ \mathcal{M} which is supported by a classical causal relation: the transitive, hence non-local (mediated), partial order. On the other hand, the quantal non-associative paradigm \mathbb{S} is a sound algebraic model of the ‘quantum causal space’ ${}_q\mathcal{M}$ which is supported by a quantal causal relation: the intransitive, hence local (immediate), succession relation (Finkelstein, 1996).

5 CONCLUSION CUM DISCUSSION

The present paper follows Trifonov’s project (1995), to model the measurement (active observation) process and express how the logic of a classical observer determines what she ‘sees’, and extends it for the ‘active observation parameters’ of a quantal researcher. The latter, we argued, builds the

²¹A local algebraic model since the paradigm \mathbb{S} is a (super)-Lie algebra (Raptis, 1998).

quantum topos $\{\$$ picture of the causal structure of quantum Minkowski spacetime ${}_q\mathcal{M}$. The subobject classifier of this non-classical linear xenomorph is an algebraic, quantum version of the Boolean binary alternative of the classical topos $\{\mathbb{H}\}$ model of classical Minkowski space \mathcal{M} and corresponds to the Lorentz-spin algebra $sl(2, \mathbb{C})$. The resulting algebraic picture of ${}_q\mathcal{M}$ as a quantum causal space was seen to agree in some manner with the recently algebraically quantized causal sets of Sorkin *et al.* (Raptis, 1999).

In a way our approach is opposite to Finkelstein's '(quantum) logics come from (quantum) dynamics' (1996) and may be squeezed to the motto '(quantum) causality comes from (quantum) logic'. Truth must lie at a synthesis of these two dual conceptions so that, as Finkelstein also points out in (1996), it is quite unnatural to separate logics from dynamics and causal structure in the quantum deep²².

We conclude this paper with a future project on an open problem. Our approach to quantum spacetime's causal structure via the quantum topos $\{\$$ is by no means complete. $\{\$$ models quantum Minkowski space that simply lacks gravity: it is globally flat. The 'true' quantum topos for spacetime should be able to accommodate some sort of 'quantum gravity'. Since gravity may be thought of as the dynamics of a variable causal connection, the force that tilts the lighcone at every event, and since a topos can be viewed as a realm of (dynamically) variable entities²³, so the quantum topos $\{\$$ may be thought of as being only locally ${}_q\mathcal{M}$, that is, be naturally 'gauged' to a twisted or curved sheaf of local paradigms $\$$. Thus it may eventually provide a theoretical scheme for describing the dynamics of the quantal lightcone (quantum causality) encoded locally in each of its stalks $\$$. As we saw in the present paper, this encodement is in the motor structure of each of its stalks $\$$, hence an abstract algebraic quantum topos scenario for quantum gravity may be expressed thus: 'quantum gravity is a theory of the dynamics of an algebraic product in a quantum topos/sheaf of locally flat ${}_q\mathcal{M}$ -algebras'²⁴.

²²For such a synthesis see end of quotation from (Finkelstein, 1996) in Section 4b'. In view of the present paper, such a synthesis may be called 'quantum causalization of the observer' and it pertains to regarding the researcher as an indiscriminable part of the deep quantum dynamical process.

²³The topos *Set* for instance can be viewed as a world of varying sets (Bell, 1988).

²⁴Since we also saw in footnote 11 of 4b' that in $\$$ there are no inverses, it also has a good chance of algebraically representing the microlocal 'quantum arrow of time' that is supposed to be the key feature of the 'true quantum gravity' (Penrose, 1987). The

The crucial question seems to be: how do we vary an algebraic product, or what amounts to the same, what is the curvature of an algebraic product?

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quantum causal nets in (Finkelstein, 1988) are time-asymmetric too, being described by chiral spinors. Also Selesnick, working on Finkelstein's quantum net, has noticed a similar asymmetry in that spacetime quanta are solely 'left-handed' by transforming exclusively under the regular representation of $SL(2, \mathbb{C})$ and not its conjugate which represents 'right-handed' quanta (Selesnick in private correspondence). Finally, in (Raptis, 1998) $\$$ is seen to be 'multiplicatively directed', that is, its generators in g participate in serial products only in a certain order resembling the normal or time-ordered products of operators in quantum field theory. On top of this asymmetry, very strong gravitational forces, such as those expected at Planck scale, should significantly tilt the lightcone so as to render causality an intransitive relation (Auyang, 1995); hence, the non-associative product in $\$$ (Section 4g') is all the more qualified to algebraically represent such a 'time-asymmetric curved causality' (Raptis, 1998).

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